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THE 2-CIRCLE AND 2-DISK PROBLEMS ON TREES

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ABSTRACT

Our purpose here is to consider on a homogeneous tree two Pompeiu-type problems which classically have been studied on the plane and on other geometric manifolds. We obtain results which have remarkably the same flavor as classical theorems. Given a homogeneous tree, let $d(x, y)$ be the distance between vertices x and y, and let f be a function on the vertices. For each vertex x and nonnegative integer n let $\Sigma_n f(x)$ be the sum $\Sigma_{d(x,y)=n} f(y)$ and let $B_n f(x) = \Sigma_{d(x,y)\leq n} f(y)$. The purpose is to study to what extent $\Sigma_n f$ and $B_n f$ determine f. Since these operators are linear, this is really the study of their kernels. It is easy to find nonzero examples for which $\Sigma_n f$ or $B_n f$ vanish for one value of n. What we do here is to study the problem for two values of n, the 2-circle and the 2-disk problems (in the cases of Σ_n and B_n respectively). We show for which pairs of values there can exist nonzero examples and we classify these examples. We employ the combinatorial techniques useful for studying trees and free groups together with some number theory.

§0. Introduction

A classical problem, with deep theoretical and practical implications, is the reconstruction of a function defined, say, on the plane, given its averages over all translates of a given set. This is the so-called "Pompeiu problem." The prototype of Pompeiu-type problems deals with circular averages and is given as follows: Let f be a function on $\mathbf{R}^2$ which is integrable over all smooth closed curves of finite length. For each $x \in \mathbf{R}^2$ and $r > 0$, let $\Sigma_r f(x)$ be the integral of f along the circle of radius r centered at x. It is easy to see that it is not possible to

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reconstruct a function f on $\mathbf{R}^2$ only from its integrals $\Sigma_r f(x)$ for one fixed r. That is, for every positive r, there exist nontrivial functions f such that $\Sigma_r f = 0$ (i.e. the zero function). However, very often we can recover the function f from its circular integrals over circles of two different radii. Indeed there is a certain set S such that if $\Sigma_r f = \Sigma_s f = 0$ for $r/s \notin S$, then $f = 0$ (The 2-Circle Theorem). S is the set of ratios of any two zeros of the Bessel function J_0. It is countable and dense in $\mathbf{R}^+$. On the other hand, if $r/s \in S$, then there are nontrivial solutions to $\Sigma_r f = \Sigma_s f = 0$. There is a similar statement concerning the functor B_r, where $B_r f(x)$ is the integral of f around the ball of radius r. In this case the set S is exactly the set of quotients of zeros of the Bessel function $J_1(z)$.

Our purpose here is to consider the two corresponding problems on a homogeneous tree. Let T be a homogeneous tree with $(q+1)$ edges at each vertex. Let $d(x, y)$ be the distance from x to y, that is, the number of edges in the path from x to y. Let $f: T \to \mathbf{C}$ be a function (i.e. on the vertices) and for each vertex x and nonnegative integer n let $\Sigma_n f(x)$ be the sum $\Sigma_{d(x,y)=n} f(y)$. Here also the sum for a single radius is not sufficient. We will study when it can happen that for f nonzero $\Sigma_n f = \Sigma_k f = 0$ for $n \neq k$, and subsequently the same problem for $B_n f = B_k f = 0$ for $n \neq k$ where

$$B_n f(x) = \sum_{d(x,y) \leq n} f(y).$$

In both cases we will discover that the exceptional set of radii is much smaller than in the case of $\mathbf{R}^2$ described above. We now state the 2-Circle Theorem:

THEOREM 1. *Let T be as above. Let $f: T \to \mathbf{C}$ be given, and let $n \neq k$ be integers.*

(a) *If n and k are odd, then $\Sigma_n f = \Sigma_k f = 0$ if and only if $\Sigma_1 f = 0$; $\Sigma_1 f = 0$ if and only if $\Sigma_{2m+1} f = 0$ for all m; $\Sigma_1 f = 0$ has nontrivial solutions.*

(b) *Assume n and k are not both odd; if q is 2 assume further that n and k are not both congruent to* 4 mod 6. *Then $\Sigma_n f = \Sigma_k f = 0$ if and only if $f = 0$.*

(c) *Assume $q = 2$ and n and k are both congruent to* 4 mod 6. *Then $\Sigma_n f = \Sigma_k f = 0$ if and only if $\Sigma_2 f = 3f$; $\Sigma_2 f = 3f$ if and only if $\Sigma_{6m+4} f = 0$ for all m; $\Sigma_2 f = 3f$ has nontrivial solutions.*

In §3 we shall construct a set of generators for the space of all functions f with $\Sigma_1 f = rf$ for any complex number r. This space contains nonzero elements. In particular putting $r = 0$, we get the generators for the exceptional functions for the case n, k both odd. We will also use this to get a set of generators for the exceptional sets for the case $q = 2$ and $\Sigma_2 f = 3f$.

The proof of Theorem 1 is based on two facts. First, we point out that the operator Σ_n can be described as a polynomial in Σ_1. In particular, then, the zeros of Σ_n are related to the roots of this polynomial. Second, by proving a result in number theory (Theorem 4) we are able to describe precisely those numbers which can be the roots of more than one such polynomial. This latter analysis is carried out in §4. Finally, in §5 we handle the corresponding problem for disks.

These problems were suggested to us by Carlos Berenstein. We would like to acknowledge useful conversations with him as well as with Wolfgang Woess and Don Zagier. The problems are related to the study of averages of summable functions over various types of sets (Radon transforms on trees; cf. [BFP] and [BCCP]) as well as averages of harmonic functions over disks [PW].

§1. Polynomials in Σ_1

Let us consider Σ_n as an operator, and let $\mathscr{A}$ be the algebra generated by the Σ_n. Notice that $\Sigma_1 \Sigma_n f(x)$ is by definition the sum of all $f(w)$ where w has distance 1 from some y which has distance n from x, taken with multiplicity if more than one y works for the same w. The distance from w to x is clearly either $n + 1$ or $n - 1$. If it is $n + 1$, then for each w there is a unique such y. If it is $n - 1$ and this is not zero, then there are exactly q such y's — all the vertices of distance 1 from w except the 1 which points toward x. On the other hand, if $n = 1$, then $w = x$, and there are $q + 1$ such y's — all the vertices of distance 1 from x. This proves the following:

Proposition 1. $\Sigma_1 \Sigma_1 = \Sigma_2 + (q + 1)\Sigma_0$, *and for* $n > 1$, $\Sigma_1 \Sigma_n = \Sigma_{n+1} + q\Sigma_{n-1}$.

Now let $p_n(x)$ be the polynomials defined by:

$$(1) \qquad \begin{gathered} p_0(x) = 1, \quad p_1(x) = x, \quad p_2(x) = x^2 - (q + 1), \\ \text{and for } n > 1, \qquad p_{n+1}(x) = xp_n(x) - qp_{n-1}(x). \end{gathered}$$

Notice that putting (1) together with Proposition 1 yields the fact that $p_n(\Sigma_1) = \Sigma_n$.

This sequence of polynomials has been studied extensively — cf. [C], where $q = 2t - 1$. Although in [C] it was assumed that t was an integer $\geqq 1$, all results and proofs remain unchanged for t a real number $\geqq 1$. We wish to study those functions f for which $\Sigma_k f = \Sigma_n f = 0$. That is the set of f for which $p_k(\Sigma_1)f = p_n(\Sigma_1)f = 0$. But since the polynomial ring $\mathbf{C}[x]$ is a principal ideal domain, there exists a unique monic polynomial $p(x)$ which generates the ideal $(p_n(x), p_k(x))$ defined as the ideal generated by $p_n(x)$ and $p_k(x)$. The two conditions then become equivalent to the single condition $p(\Sigma_1)f = 0$. In the case that $p_n(x)$ and $p_k(x)$ are relatively prime, $p(x) = 1$ and the condition implies that $f = 0$. Since $\mathbf{C}$ is algebraically closed, $p_n(x)$ and $p_k(x)$ are relatively prime if and only if they have no common roots. Note also that if r is a common root of $p_n(x)$ and $p_k(x)$, then $\Sigma_n f$ and $\Sigma_k f$ both have $(\Sigma_1 f - rf)$ as a common factor. If $f \neq 0$ and $\Sigma_1 f = rf$, then we get an example of a nonzero f with $\Sigma_n f = \Sigma_k f = 0$. Thus it makes sense to study the common roots of the polynomials $p_n(x)$. Theorem 1 — except for the eigenfunctions which are to be constructed in §3 — is equivalent to the following:

THEOREM 2. (a) *If n and k are odd then* $(p_n(x), p_k(x)) = (x) = (p_{2m+1}(x))_{m \in \mathbf{N}}$.

(b) *Assume n and k are not both odd; if q is* 2 *assume further that n and k are not both congruent to* 4 mod 6. *Then* $(p_n(x), p_k(x)) = (1)$.

(c) *Assume $q = 2$ and n and k are both congruent to* 4 mod 6. *Then* $(p_n(x), p_k(x)) = (x^2 - 6) = (p_2(x) - 3) = (p_{6m+4}(x))_{m \in \mathbf{N}}$.

§2. Proof of the 2-circle theorem

Notice that the relation $x = 2\omega \cos\theta$ sets up a homeomorphism between the interval $[-2\omega, 2\omega]$ and $[0, \pi]$ (oriented negatively) for any positive real number ω. Let us consider the functions $p_n(x)$ as defined on $[0, \pi]$. They satisfy

$$p_0(x) = 1, \quad p_1(x) = 2\omega\cos\theta,$$

$$p_2(x) = 4\omega^2\cos^2\theta - (q+1) = \omega^2\left\{\frac{2\cos\theta\sin 2\theta}{\sin\theta} - \frac{q+1}{\omega^2}\,\frac{\sin\theta}{\sin\theta}\right\} \tag{2}$$

$$\text{and for } n > 1, \qquad p_{n+1}(x) = 2\omega\cos\theta\, p_n(x) - q p_{n-1}(x).$$

Using (2) with $\omega^2 = q$, one gets the general formula

$$p_n(x) = \omega^n\left\{\frac{\sin(n+1)\theta}{\sin\theta} - \frac{1}{q}\,\frac{\sin(n-1)\theta}{\sin\theta}\right\}. \tag{3}$$

In particular notice that this yields

under the relations $x = 2\omega \cos\theta$ and $\omega^2 = q$,

(4) $$p_n(x) = 0 \text{ if and only if } q\sin(n+1)\theta = \sin(n-1)\theta, \quad \theta > 0.$$

Next we need to know the following result, suggested to us by Don Zagier. There is a proof in §4.

PROPOSITION 2. *Assume* $\alpha, \beta \in \mathbf{Q}\pi$, *and* $(\sin\alpha)/(\sin\beta) = r \in \mathbf{Q}$, $r > 1$. *Then* $\sin\alpha = \pm 1$ *and* $\sin\beta = \pm 1/2$.

Assume that $\theta \notin \mathbf{Q}\pi$ and $p_n(x) = 0$. Notice that $\sin(n+1)\theta = (1/q)\sin(n-1)\theta$, thus

$$\cos 2\theta \sin(n-1)\theta + \sin 2\theta \cos(n-1)\theta = (1/q)\sin(n-1)\theta.$$

Since $\theta \notin \mathbf{Q}\pi$, $\cos(n-1)\theta$ and $\sin 2\theta$ are nonzero, and so $\cos 2\theta \neq 1/q$. Thus we get

$$\tan(n-1)\theta = \sin 2\theta/(1/q - \cos 2\theta).$$

Assume that $p_k(x) = 0$ also. Then

$$\tan(k-1)\theta = \sin 2\theta/(1/q - \cos 2\theta),$$

so that $\tan(n-1)\theta = \tan(k-1)\theta$ whence $(k-1)\theta = (n-1)\theta + m\pi$ contradicting the fact that $\theta \notin \mathbf{Q}\pi$ unless $n = k$. Thus $\theta \in \mathbf{Q}\pi$ if it is a root of $p_n(x)$ and $p_k(x)$ for $n \neq k$.

Next note that $\sin\theta = 0$ corresponds to $x = \pm 2\omega$, or $\theta = 0, \pi$. By taking limits in (3) we see that

$$p_n(\pm 2\omega) = \pm\omega^n\{n + 1 - (n-1)/q\}$$

which is never 0. Thus no zero of $p_n(x)$ can correspond to the case $\sin\theta = 0$.

So now let us study zeros of $p_n(x)$ where $\theta \in \mathbf{Q}\pi$. We know that $\sin\theta \neq 0$ and that $q\sin(n+1)\theta = \sin(n-1)\theta$. If $\sin(n+1)\theta = \sin(n-1)\theta = 0$, we get $\cos\theta = 0$, whence $\theta = \pi/2$ and $x = 0$. This implies that $p_n(0) = 0$ for all n odd. This shows that $(x) \subseteq (p_{2m+1}(x))_{m\in\mathbf{N}}$, and that any two odd-degree polynomials in our set have 0 as a common root. If it is not the case that $\sin(n+1)\theta = \sin(n-1)\theta = 0$, we may apply Proposition 2 to the equation $\sin(n-1)\theta = q\sin(n+1)\theta$ and we see that q must be 2 and $\sin(n+1)\theta = \pm\frac{1}{2}$.

Now we need to study the case $q = 2$, $\sin(n+1)\theta = \pm\frac{1}{2}$ and $\sin(n-1)\theta = \pm 1$. We look first at the case $\sin(n+1)\theta = +\frac{1}{2}$ and $\sin(n-1)\theta = +1$. Thus for some integers a and b,

$$(n+1)\theta = (2a + \tfrac{1}{2} \pm \tfrac{1}{3})\pi \qquad \text{and} \qquad (n-1)\theta = (2b + \tfrac{1}{2})\pi.$$

Thus $\theta = \{(a-b) \pm \frac{1}{6}\}\pi$, but $\theta \in (0, \pi)$, so $\theta = (3 \mp 2)\pi/6$. Thus $(n+1)(3 \mp 2)\pi/6 = (2a + \frac{1}{2} \pm \frac{1}{3})\pi$, whence $n = 12a + 4$ or $5n = 12a - 4$, and in either case $n \equiv 4 \pmod{12}$. Now we look at the case $\sin(n+1)\theta = -\frac{1}{2}$ and $\sin(n-1)\theta = -1$. Thus for some integers a and b,

$$(n+1)\theta = (2a - \tfrac{1}{2} \pm \tfrac{1}{3})\pi \quad \text{and} \quad (n-1)\theta = (2b - \tfrac{1}{2})\pi.$$

Thus $\theta = \{(a-b) \pm \frac{1}{6}\}\pi$, but $\theta \in (0, \pi)$, so $\theta = (3 \mp 2)\pi/6$. Thus $(n+1)(3 \mp 2)\pi/6 = (2a - \frac{1}{2} \pm \frac{1}{3})\pi$, whence $n = 12(a/5) - 2$ or $n = 12a - 2$ and thus $n \equiv 10 \pmod{12}$. Thus we have the result that $p_n(x)$ and $p_k(x)$ can have a common zero only for $q = 2$ and $n \equiv 4 \pmod 6$.

Since the $p_n(x)$ are even functions for n even, for some number r they will have a common factor of $(x^2 - r^2)$. We also know that the common roots should correspond to $\theta = \pi/6$ and $5\pi/6$, whence $x = 2\sqrt{2}\cos\theta = \pm\sqrt{6}$, and the common factor is $(x^2 - 6) = p_2(x) - 3$. For the other direction, observe using (1) that $p_4(x) = (x^2 - 4)\,p_2(x) - 6$, and for $n > 2$,

$$p_{n+2}(x) = (x^2 - 4)\,p_n(x) - 4p_{n-2}(x).$$

Thus if $x = \pm\sqrt{6}$, then we have $p_2(x) = 3$, $p_4(x) = 0$, $p_6(x) = -12$, $p_8(x) = -24$, $p_{10}(x) = 0$ and so by induction we get that $p_{6m+4}(x) = 0$. This completes the proof of the theorem.

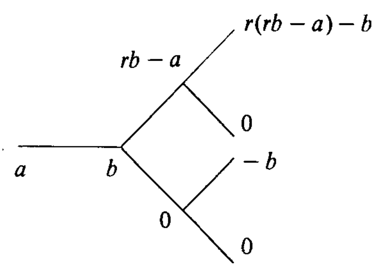
§3. Eigenfunctions of Σ_1

In this section we shall describe the eigenfunctions of Σ_1. Let T be a given tree. Remove an edge $E = [v_1, v_2]$ from T, but not the vertices. This leaves two components, T_1 and T_2 containing the vertices v_1 and v_2, respectively. A *branch* is the union of E with one of these components. We then have the branches $B_1 = T_1 \cup E$ and $B_2 = T_2 \cup E$. $T = T_1 \cup T_2 \cup E$, and we say that T is gotten by *grafting* T_1 to T_2. Call v_1 the *end point* of T_2 and v_2 the end point of T_1. We shall refer to v_1 and v_2 as the second points of T_1 and T_2 respectively. We will also say that T_1 is the branch with leading edge $[v_2, v_1]$ and T_2 is the branch with leading edge $[v_1, v_2]$. For now we assume that $q > 1$. Later we shall consider separately the more trivial case of $q = 1$.

A function on a tree may be described by assigning values to the vertices. Let r be an arbitrary fixed complex number. We shall now describe a set of generators for the eigenfunctions of Σ_1 of eigenvalue r. We shall define a certain type of function on a branch inductively on the distance from the end point. We will say that a function on a branch B is (a, b) elementary (with respect to the fixed eigenvalue r) if it can be described as follows: to the endpoint v_1 is assigned the value a and to the second point v_2 is assigned the value b. Let $w_1, w_2, \ldots, w_q$ be the vertices attached to the second point, other than the end point. Choose one of them, say w_1, and assign to it the value $(rb - a)$ and to the others assign the value 0. To the branch with leading edge $[v_2, w_1]$ we will assign a $(b, rb - a)$ elementary function and to the branch with leading edge $[v_2, w_i]$ for $i > 1$ we will assign a $(b, 0)$ elementary function. Obviously we need the axiom of choice to insure the existence of (a, b) elementary functions on branches. Let f be an (a, b) elementary function so defined. Notice that

$$\Sigma_1 f(v_2) = a + (rb - a) + 0 + \cdots + 0 = rb = rf(v_2).$$

So by induction $\Sigma_1 f(v) = rf(v)$ for every vertex of B except the endpoint.

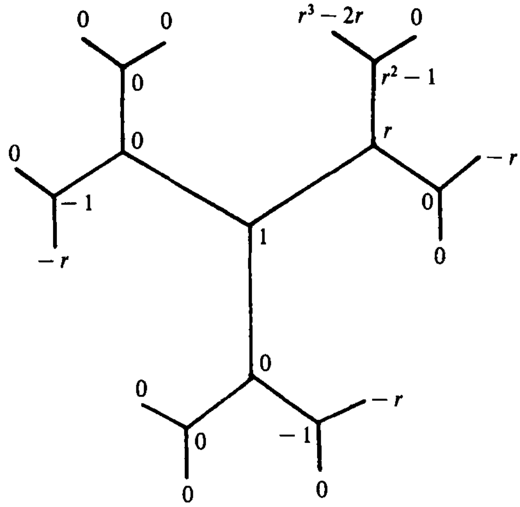


We shall now define an elementary eigenfunction f on a tree. Choose a vertex v which is attached to vertices $w_1, w_2 \ldots, w_{q+1}$. Assign to v the value 1, to w_1 the value r and to $w_2, \ldots, w_{q+1}$ the values 0. To the branch with leading edge $[v, w_1]$ we will assign a $(1, r)$ elementary function and to the branch with leading edge $[v, w_i]$ for $i > 1$ we will assign a (1,0) elementary function. By the remarks above, $\Sigma_1 f(w) = rf(w)$ for every vertex w except possibly v itself. But by choice

$$\Sigma_1 f(v) = \sum f(w_i) = r = rf(v).$$

Thus f is an eigenfunction of Σ_1 with eigenvalue r. We will say that f is an elementary function about the edge $[v, w_1]$. If more precision is required, we will say that f is an elementary function about the edges $[v, w_1, u]$ where u is the vertex connected to w_1 at which f takes on the value $r^2 - 1$.

We shall now show that every eigenfunction can be written as a linear combination of elementary functions, possibly infinite but locally finite. Let g be an eigenfunction. Choose a vertex v which is attached to vertices $w_1, w_2, \ldots, w_{q+1}$. Let $g(v) = a$ and $g(w_i) = b_i$. $\Sigma\, b_i = ra$. Let f_i be an elementary



function about the edge $[v, w_i]$, and for $i > 1$, let f_i' be an elementary function about the edges $[w_i, v, w_1]$. For $r \neq 0$, let

$$h = g - \frac{1}{r} \sum b_i f_i$$

and for $r = 0$, let

$$h = g - \sum b_i f_i'.$$

Notice that it is still an eigenfunction, but its value is 0 at $v, w_1, w_2, \dots, w_{q+1}$. Let $u_1, u_2 \dots, u_q$ be the vertices other than v attached to w_1. Then we subtract $f(u_1)k$ from h where k is an elementary function about the edges $[u_1, w_1, u_2]$. Notice that this does not change any value at vertices on the branch with leading edge $[w_1, v]$, only those on the branch with leading edge $[v, w_1]$. We can continue in this manner to subtract off linear combinations of elementary functions, each subtraction having no effect on what has happened before. We can keep extending the circle where all values are zero, changing the value at each vertex only a finite number of times. Thus we get the following.

THEOREM 3. *Every eigenfunction can be expressed as a locally finite linear combination of elementary functions.*

Putting $r = 0$ give us the set of solutions to Theorem 1(a). Now for Theorem 1(c), let $f^{\pm}$ be eigenfunctions of Σ_1 for the eigenvalues $r = \pm\sqrt{6}$. Since $\Sigma_1 f^{\pm} = \pm\sqrt{6} f^{\pm}$, $\Sigma_2 f^{\pm} = \Sigma_1^2 f^{\pm} - 3f^{\pm} = 6f^{\pm} - 3f^{\pm} = 3f^{\pm}$. Thus any linear combination g of f^+ and f^- would also satisfy $\Sigma_2 g = 3g$. Conversely, if $\Sigma_2 g = 3g$, then let

$$f^{\pm} = \left[\frac{\Sigma_1 \pm \sqrt{6}}{2\sqrt{6}}\right] g.$$

Notice that $\Sigma_1 f^{\pm} = \pm\sqrt{6}\, f$ and that $g = f^+ - f^-$.

Now let us consider the case $q = 1$. In this case we can represent the vertices as v_n and the edges as $[v_n, v_{n+1}]$ for $n \in \mathbf{Z}$. Let f be the function defined by

$$f(v_n) = \frac{1}{2^n\sqrt{r^2-4}}\{(r+\sqrt{r^2-4}))^n - (r-\sqrt{r^2-4}))^n\} \qquad \text{for } n \geqq 0,$$

and

$$f(v_n) = -f(v_{-n}) \qquad \text{for } n < 0;$$

this is for $r \neq 2$. For $r = 2$, let $f(v_n) = n$. Note that $f(v_0) = 0, f(v_1) = 1, f(v_2) = r$, $f(v_3) = r^2 - 1$, etc. The elementary functions are the functions f and f' where f' is given by $f'(v_n) = f(v_{1-n})$. Then $f'(v_0) = 1$ and $f'(v_1) = 0$. Assume that g is an eigenfunction of Σ_1. Let

$$h = g - g(v_0)f' - g(v_1)f.$$

Then h is an eigenfunction of Σ_1 such that $h(v_0) = h(v_1) = 0$. Since

$$rh(v_n) = h(v_{n+1}) + h(v_{n-1}),$$

it is easy to see that $h = 0$. Thus $g = g(v_0)f' + g(v_1)f$.

§4. A divisibility theorem

This section is dedicated to proving Proposition 2. The proof is based on the following divisibility result:

THEOREM 4. *Let $\rho \neq 1$ be a primitive root of unity and $k > 1$ an integer.*

(1) *Assume that k divides $(1-\rho)$ in some finitely generated ring of algebraic integers. Then $k = 2$ and $\rho = -1$.*

(2) *Assume that k divides $(1-\rho)^2$ in some finitely generated ring of algebraic integers. Then $k = 2$ and $\rho = -1, \pm i$, or else $k = 3$ and $\rho = (-1 \pm i\sqrt{3})/2$, or else $k = 4$ and $\rho = -1$.*

PROOF. For an integer $n \geqq 1$, let $\Phi_n(x)$ be the cyclotomic polynomial $\Pi(x-\lambda)$, where the product is taken over all primitive nth roots of unity. Using the fact that

$$\Phi_n(x) = \frac{1 - x^n}{\prod\limits_{\substack{d \mid n \\ d < n}} \Phi_d(x)} = \frac{1 + x + \cdots + x^{n-1}}{\prod\limits_{\substack{d \mid n \\ 1 < d < n}} \Phi_d(x)},$$

it follows that

$$\Phi_n(1) = \frac{n}{\prod\limits_{\substack{d \mid n \\ 1 < d < n}} \Phi_d(1)}$$

and so one can prove easily by induction that $\Phi_n(1) = p$ if n is a power of the prime p, and is 1 if n is not a prime power. Assume that k divides $(1 - \rho)^a$ where ρ is one primitive nth root of unity. If λ is any other primitive nth root of unity, then $\lambda = \rho^s$ for some positive integer s, so that $(1 - \rho)$ divides $(1 - \lambda)$, whence k divides $(1 - \lambda)^a$. Thus letting $\varphi = \varphi(n)$, the number of primitive nth roots of unity, we see that k^φ divides $\Phi_n(1)^a$. Hence for some prime p and some integer $t \geqq 1$, $n = p^t$, with k^φ dividing p^a. Thus $k = p^s$, $\varphi = (p - 1)p^{t-1}$, and $s\varphi \leqq a$. For the case of $a = 1$, this means that $s = \varphi = 1$, and so $n = 2$, $p = 2$, $k = 2$, and $\rho = -1$, proving (1). For the case of $a = 2$, we have $s\varphi = 1$ or 2. If $s\varphi = 1$, then $s = \varphi = 1$ and so $n = 2$, $p = 2$, $k = 2$, and $\rho = -1$. Assume that $s\varphi = 2$. If $s = 2$, then $\varphi = 1$, so $n = 2$, $p = 2$, $k = 4$, and $\rho = -1$. But if $s = 1$, then $\varphi = 2$, so that $n = 3$ or 4. If $n = 3$, then $p = 3$, $k = 3$, and $\rho = (-1 \pm i\sqrt{3})/2$. If $n = 4$, then $p = 2$, $k = 2$, and $\rho = \pm i$. This proves (2). ■

Returning to the proof of Proposition 2,

$$\sin\alpha = \frac{\lambda - \lambda^{-1}}{2i} \quad \text{and} \quad \sin\beta = \frac{\mu - \mu^{-1}}{2i},$$

where λ and μ are roots of unity. Writing $r = s/t$, where s and t are positive relatively prime integers with $s > 1$, we get

$$t\lambda^{-1}(1 - \lambda^2) = s\mu^{-1}(1 - \mu^2).$$

Since λ and μ are units, we conclude that $t \mid (1 - \mu^2)$ and $s \mid (1 - \lambda^2)$. Thus $s = 2$, $\lambda^2 = -1$, and $t = 1$. So $r = 2$, $\lambda = \pm i$, whence $\sin\alpha = \pm 1$, and so $\sin\beta = \pm\frac{1}{2}$. ■

§5. The problem of disks

We now consider a similar problem on the homogeneous tree T, but instead of adding up over the *circle* of radius n, we add up over the *disk* of radius n. That is, we consider the operators B_n defined on the function $f\colon T \to \mathbf{C}$ by

$$B_n f(x) = \sum_{d(x,y) \leq n} f(y).$$

It is clear that the operators B_n may be expressed in terms of the Σ_n by the formula

$$B_n = \sum_{k=0}^{n} (\Sigma_k).$$

We thus observe that

$$\Sigma_1 B_n = \Sigma_1 \Sigma_0 + \Sigma_1 \Sigma_1 + \sum_{k=2}^{n} (\Sigma_1 \Sigma_k) = \Sigma_1 + \Sigma_2 + (q+1)\Sigma_0$$

$$+ \sum_{k=2}^{n} (\Sigma_{k+1} + q\Sigma_{k-1}).$$

Notice that in both the $n = 1$ and the $n > 1$ cases, this yields $B_{n+1} + qB_{n-1}$. Thus if we define a sequence of polynomials by

$$r_0(x) = 1, \quad r_1(x) = x + 1 \quad \text{and} \quad xr_n(x) = r_{n+1}(x) + qr_{n-1}(x),$$

we find that $r_n(\Sigma_1) = B_n$.

We ask when $B_n f = B_k f = 0$ implies that $f = 0$. We can answer the question, just as in the case of Σ_n and Σ_k, by looking for common zeros of the polynomials $r_n(x)$. Let $\omega = \sqrt{q}$ as before, and define

$$t_n(x) = \omega^{-n} r_n(2\omega x);$$

we find that $t_n(x)$ satisfies:

$$t_0(x) = 1, \quad t_1(x) = 2x + \omega^{-1} \quad \text{and} \quad xt_n(x) = t_{n+1}(x) + t_{n-1}(x).$$

The Čebyšev polynomials $U_n(x)$ defined by

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}$$

satisfy the same recursive formula with $U_0(x) = 1$ and $U_1(x) = 2x$. Thus we have

$$t_n(x) = U_n(x) + \omega^{-1} U_{n-1}(x).$$

Now a zero of $r_n(x)$ corresponds to a zero of $t_n(x)$, and therefore corresponds to a zero of

$$\frac{\sin(n+1)\theta}{\sin\theta}+\omega^{-1}\frac{\sin(n\theta)}{\sin\theta},$$

and thus to a zero of $\sin(n+1)\theta+\omega^{-1}\sin(n\theta)$.

So assume that

$$\sin(n+1)\theta+\omega^{-1}\sin(n\theta)=\sin(k+1)\theta+\omega^{-1}\sin(k\theta)=0, \qquad \text{for } n\neq k.$$

The first equation yields, as in the case of the polynomials $p_n(x)$, that $\theta\in\mathbf{Q}\pi$. Thus there remains only the problem:

Find the solutions to $\sin(n+1)\theta+\omega^{-1}\sin(n\theta)=0$, $\theta\in\mathbf{Q}\pi$.

We again use Theorem 4 to prove a result similar to Proposition 2.

Proposition 3. *Assume $\alpha,\beta\in\mathbf{Q}\pi$, and*

$$\frac{\sin\alpha}{\sin\beta}=\sqrt{r}, \qquad r\in\mathbf{Z}, \quad r>1.$$

Then $r=2,3,$ or 4. If $r=2$, $\sin\alpha=\pm 1,\pm\sqrt{2}$ and $\sin\beta=\pm 1/\sqrt{2},\pm 1$. If $r=3$, $\sin\alpha=\pm\sqrt{3}/2$ and $\sin\beta=\pm\frac{1}{2}$. If $r=4$, then $\sin\alpha=\pm 1$ and $\sin\beta=\pm\frac{1}{2}$.

Proof. As before we let

$$\sin\alpha=\frac{\lambda-\lambda^{-1}}{2i} \qquad \text{and} \qquad \sin\beta=\frac{\mu-\mu^{-1}}{2i},$$

where λ and μ are roots of unity. We get

$$\lambda^{-2}(1-\lambda^2)^2=r\mu^{-2}(1-\mu^2)^2.$$

Since λ and μ are units, we conclude that $r\mid(1-\mu^2)^2$. Now by Theorem 4(2), we get that $r=2,3$ or 4. By that result, when $r=2$, we get $\lambda^2=-1,\pm i$. Thus $2\alpha\equiv\pi,\pm\pi/2 \pmod{2\pi}$, or $\alpha\equiv\pi/2,\pm\pi/4 \pmod{\pi}$, yielding the results $\sin\alpha=\pm 1,\pm\sqrt{2}$, whence $\sin\beta=\pm 1/\sqrt{2}$, ± 1. When $r=3$, $\lambda^2=(-1\pm 1/\sqrt{2})/2$, so $2\alpha\equiv 2\pi/3, 4\pi/3 \pmod{2\pi}$, i.e. $\alpha\equiv\pi/3, 2\pi/3 \pmod{\pi}$, yielding the above result. And finally the case $r=4$ corresponds to the case $r=2$ of Proposition 2. ∎

We can now solve the problem of finding the values of q and n such that

$$\sin(n+1)\theta + \omega^{-1}\sin(n\theta) = 0, \qquad \theta \in \mathbf{Q}\pi \cap [0, \pi].$$

That is, we need to solve

$$\frac{\sin(n\theta)}{\sin(n+1)\theta} = -\sqrt{q}.$$

This is Proposition 3 with $\alpha = n\theta$ and $\beta = -(n+1)\theta$. So we know that $q = 2, 3$ or 4.

First for $q = 2$: $\sin(n\theta) = \pm 1, \pm\sqrt{2}/2$, $\sin(n+1)\theta = \mp 1/\sqrt{2}, \mp \frac{1}{2}$. We take the pair $(1, -1/\sqrt{2})$. Thus $n\theta \equiv \pi/2 \pmod{2\pi}$ and $(n+1)\theta \equiv 5\pi/4, 7\pi/4 \pmod{2\pi}$. Thus $\theta \equiv 3\pi/4, 5\pi/4 \pmod{2\pi}$. But this second is not allowed since $\theta \in [0, \pi]$. Thus $\theta = 3\pi/4$, and $3n\pi/4 \equiv \pi/2 \pmod{2\pi}$, yielding $3n \equiv 2 \pmod 8$ or $n \equiv 6 \pmod 8$. In a similar manner, the pair $(-1, 1/\sqrt{2})$ also yields $\theta = 3\pi/4$, but with $n \equiv 2 \pmod 8$. Thus we find a common root, $x = 2\sqrt{2}\cos(3\pi/4) = -2$, for all $r_n(x)$ where $n \equiv \pm 2 \pmod 8$.

Next look at the pair $(\sqrt{2}/2, -\frac{1}{2})$. $n\theta \equiv \pi/4, 3\pi/4 \pmod{2\pi}$ and $(n+1)\theta \equiv 7\pi/6, 11\pi/6 \pmod{2\pi}$. Recalling that $\theta \in [0, \pi]$, we get $\theta \equiv 11\pi/12$ for $n\theta \equiv \pi/4 \pmod{2\pi}$ and $\theta \equiv 5\pi/12$ for $n\theta \equiv 3\pi/4 \pmod{2\pi}$. The first solution yields $n \equiv 9 \pmod{24}$ and the second yields $n \equiv 21 \pmod{24}$. If we now consider the case $(-\sqrt{2}/2, \frac{1}{2})$, we find the solutions $\theta \equiv 5\pi/12$ for $n \equiv 9 \pmod{24}$ and $\theta \equiv 11\pi/12$ for $n \equiv 21 \pmod{24}$. Thus we get that $\theta = 5\pi/12$ and $11\pi/12$ are zeros for $n \equiv 9 \pmod{12}$. This corresponds to the roots $-1 \pm \sqrt{3}$.

Next we take the case $q = 3$: $\sin(n\theta) = \pm\sqrt{3}/2$, $\sin(n+1)\theta = \mp\frac{1}{2}$. Consider first the pair $(\sqrt{3}/2, -\frac{1}{2})$. So $n\theta \equiv \pi/3, 4\pi/3 \pmod{2\pi}$ and $(n+1)\theta \equiv 7\pi/6, 11\pi/6 \pmod{2\pi}$. The allowable solutions then are $\theta = 5\pi/6$ for $n\theta \equiv \pi/3 \pmod{2\pi}$ and $\theta = \pi/2$ for $n\theta \equiv 4\pi/3 \pmod{2\pi}$. But notice that the latter requires solving $3n \equiv 8 \pmod{12}$ which has no solutions. The first yields $5n \equiv 2 \pmod{12}$ or $n \equiv 10 \pmod{12}$. Similarly the pair $(-\sqrt{3}/2, \frac{1}{2})$ yields only the solution $\theta = 5\pi/6$ and $n \equiv 2 \pmod{12}$. Thus we get the root $x = 2\sqrt{3}\cos(5\pi/6) = -3$ for $n \equiv \pm 2 \pmod{12}$.

Finally we consider the case $q = 4$: $\sin(n\theta) = \pm 1$, $\sin(n+1)\theta = \mp\frac{1}{2}$. Taking first the pair $(1, -\frac{1}{2})$, we get $n\theta \equiv \pi/2 \pmod{2\pi}$ and $(n+1)\theta \equiv 7\pi/6, 11\pi/6 \pmod{2\pi}$. This yields only the solution $\theta = 2\pi/3$, but then $2n\pi/3 \equiv \pi/2 \pmod{2\pi}$ is equivalent to $4n \equiv 3 \pmod{12}$ which has no integral solutions. Similarly the pair $(-1, \frac{1}{2})$ yields no solutions.

This proves the following result:

THEOREM 5. *The common zeros of the sequence* $r_n(x)$ *are given by the following information*:

(1) *If* $q = 2$, *then* $(x + 2)$ *is a common factor of all* $r_n(x)$ *where* $n \equiv \pm 2 \pmod 8$ *and* $(x^2 + 2x - 2)$ *is a common factor of all* $r_n(x)$ *where* $n \equiv 9 \pmod{12}$.

(2) *If* $q = 3$, *then* $(x + 3)$ *is a common factor of all* $r_n(x)$ *where* $n \equiv \pm 2 \pmod{12}$.

These are the only common factors possible.

Applying this directly to the operators B_n we now get

THEOREM 6. *Let* $f: T \to \mathbf{C}$ *be given, and let* $n \neq k$ *be integers.*

(a) *Assume that* $q > 3$, *or that* $q = 3$ *but* n *and* k *are not both* $\equiv \pm 2 \pmod{12}$, *or that* $q = 2$ *but* n *and* k *are not both* $\equiv \pm 2 \pmod 8$ *or both* $\equiv 9 \pmod{12}$. *Then* $B_n f = B_k f = 0$ *if and only if* $f = 0$.

(b) *Assume that* $q = 3$ *and* $\pm n \equiv \pm k \equiv 2 \pmod{12}$. *Then* $B_n f = B_k f = 0$ *if and only if* $\Sigma_1 f = -3f$; $\Sigma_1 f = -3f$ *if and only if* $B_{12m \pm 2} f = 0$ *for all* m; $\Sigma_1 f = -3f$ *has nontrivial solutions.*

(c) *Assume that* $q = 2$ *and* $\pm n \equiv \pm k \equiv 2 \pmod 8$. *Then* $B_n f = B_k f = 0$ *if and only if* $\Sigma_1 f = -2f$; $\Sigma_1 f = -2f$ *if and only if* $B_{8m \pm 2} f = 0$ *for all* m; $\Sigma_1 f = -2f$ *has nontrivial solutions.*

(d) *Assume that* $q = 2$ *and* $n \equiv k \equiv 9 \pmod{12}$. *Then* $B_n f = B_k f = 0$ *if and only if* $B_2 f + B_1 f = f$; $B_2 f + B_1 f = f$ *if and only if* $B_{12m+9} f = 0$ *for all* m; $B_2 f + B_1 f = f$ *has nontrivial solutions.*

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